

# Quantum Loop Subalgebra and Eigenvectors of the Superintegrable Chiral Potts Transfer Matrices

Helen Au-Yang and Jacques H H Perk

Department of Physics, Oklahoma State University, 145 Physical Sciences, Stillwater, OK 74078-3072, USA

Mathematical Institute of Science, Department of Theoretical Physics, Australian National University, Canberra, Australia

E-mail: perk@okstate.edu, helenperk@yahoo.com

**Abstract.** It was shown that for  $Q = 0$  and  $L$  a multiple of  $N$ , the ground state sector eigenspace of the superintegrable  $\tau_2(t_q)$  model is highly degenerate, and is generated by a quantum loop algebra  $L(\mathfrak{sl}_2)$ . Furthermore, this loop algebra can be decomposed into  $r = (N-1)L/N$  simple  $\mathfrak{sl}_2$  algebras. For  $Q \neq 0$ , we shall show that this eigenspace of  $\tau_2(t_q)$  is still highly degenerate, but split into two spaces, each containing  $2^{r-1}$  eigenvectors. The generators for the  $\mathfrak{sl}_2$  subalgebra and also for the quantum loop subalgebra, which are generalizations of those in the  $Q = 0$  case are given here. However, the Serre relations for the generators of the loop subalgebra are only proven for some states, tested on small systems and conjectured otherwise.

## 1. Introduction

Baxter [1, 2, 3, 4] has obtained the  $2^{m_Q}$  eigenvalues of the transfer matrix of the  $N$ -state superintegrable chiral Potts model, where  $m_Q = \lfloor L(N-1)/N - Q/N \rfloor$ , in terms of the roots of the Drinfeld polynomial

$$P_Q(z) = N^{-1} t^{-Q} \sum_{a=0}^{N-1} \omega^{-Qa} \frac{(1-t^N)^L}{(1-\omega^a t)^L} = \sum_{m=0}^{m_Q} \Lambda_m^Q z^m = \Lambda_{m_Q}^Q \prod_{j=1}^{m_Q} (z - z_{j,Q}). \quad (1)$$

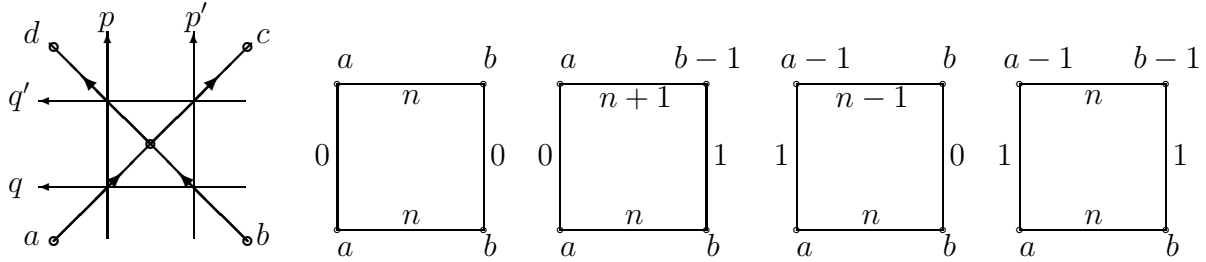
For  $0 \leq Q \leq N-1$ ,  $\omega^Q$  denotes the eigenvalue of the spin shift operator  $\mathcal{X}$ .

For  $Q = 0$  and  $L$  a multiple of  $N$ , it has been shown [5, 6] that the ground state sector eigenspace of  $\tau_2(t_q)$  is highly degenerate, and supports a quantum loop algebra  $L(\mathfrak{sl}_2)$ . Furthermore, this loop algebra can be decomposed into  $r = m_0$  simple  $\mathfrak{sl}_2$  algebras. These results enabled us to express the transfer matrix in terms of the generators of  $r$   $\mathfrak{sl}_2$  algebras [7], so that the corresponding  $2^r$  eigenvectors of the transfer matrix were found, where  $r = m_0 = L(N-1)/N$ .

For  $Q \neq 0$  cases, some investigation for the six-vertex model at root of unity was done in [9], except that not much was known. However, as the eigenvalues of transfer matrix have exactly the same property for  $Q = 0$  as well as for  $Q \neq 0$ , this gave us confidence that it must work out for  $Q \neq 0$ . Here we report the progress that has been

made. We generalize many of the results that we obtained in [6, 7] for  $Q = 0$  to  $Q \neq 0$  cases by first checking these results on a computer for small  $N$  and  $L$  and then proving them analytically. Before proceeding, we will show the differences in our notations with those of Baxter for the  $\tau_2(t_q)$  model [10], and the differences with the work of Nishino and Deguchi [5].

### 1.1. Preliminaries



**Figure 1.** The star weight and the four nonvanishing square weights for  $\tau_2(t_q)$

We consider as in [11] a star consisting of four chiral Potts weights, shown in Fig. 1,

$$U_{p'pq'q}(a, b, c, d) \equiv \sum_{e=1}^N W_{pq}(a-e) \overline{W}_{pq'}(e-d) \overline{W}_{p'q}(b-e) W_{p'q'}(e-c). \quad (2)$$

For the case  $\{x_{q'}, y_{q'}, \mu_{q'}\} = \{y_q, \omega^2 x_q, \mu_q^{-1}\}$ , it was shown in [11] that

$$U_{p'pq'q}(a, b, c, d) = 0 \quad \text{for } 0 \leq \alpha \leq 1 \text{ and } 2 \leq \beta \leq N-1; \quad \alpha \equiv a-d, \beta \equiv b-c. \quad (3)$$

The product of two transfer matrices becomes a direct sum of  $\tau_2(t_q)$  and  $\tau_{N-2}(t_q)$ , where the four nonvanishing configurations of  $\tau_2(t_q)$  are shown in Fig. 1. We have  $U_{p'pq'q}(a, b, c, d) \rightarrow C_{p'pq} U_{p'pq}^{(2)}(a, b, c, d)$ , with  $C_{p'pq}$  some constant given in [11], and

$$U_{p'pq}^{(2)}(a, b, c, d) = \mu_p^\alpha \mu_{p'}^\beta \left[ \left( \frac{1}{y_p} \right)^\alpha \left( -\frac{\omega t_q}{y_{p'}} \right)^\beta + \omega^{d-b} \left( -\frac{\omega t_q}{y_p y_{p'}} \right) \left( \frac{x_p}{t_q} \right)^\alpha \left( -\omega x_{p'} \right)^\beta \right], \quad (4)$$

which is related to equation (14) of Baxter [10] by

$$W_\tau(t_q|a, b, c, d) = (-\omega t_q)^{a-d-b+c} U_{p'pq}^{(2)}(a, b, c, d). \quad (5)$$

The factor in front cancels out upon multiplying adjacent squares together, leaving  $\tau_2(t_q)$  the same. Replacing  $p, p'$  by  $r, r'$  in (4), and letting  $\{x_{r'}, y_{r'}, \mu_{r'}\} = \{y_r, \omega^2 x_r, \mu_r^{-1}\}$  we find that the square is nonzero for  $0 \leq d-c, a-b \leq 1$ , and the nonzero elements in (4) become proportional to weights of a six-vertex model, namely

$$\left( \frac{\mu_r}{y_r} \right)^{\beta-\alpha} U_{r'rq}^{(2)}(a, b, c, d) \rightarrow U_{rq}^{(2,2)}(a, b, c, d) = \left( -\frac{t_q}{\omega t_r} \right)^\beta - (-1)^\beta \omega^{d-c-1} \left( \frac{t_q}{t_r} \right)^{1-\alpha}, \quad (6)$$

which is related to equation (6) of Baxter in [10] by

$$W_{6v}(t_r/t_q|a, b, c, d) = (\omega t_r/t_q)(t_q/t_r)^{b-a-c+d} U_{rq}^{(2,2)}(b, c, d, a), \quad (7)$$

in which the vertices are cyclicly permuted.

Consequently, the Yang–Baxter equation of the chiral Potts model becomes the Yang–Baxter equation for these squares

$$\begin{aligned} \sum_{d=1}^N U_{p'pr}^{(2)}(a, g, e, f) U_{p'pq}^{(2)}(b, c, g, a) U_{rq}^{(2,2)}(c, d, e, g) \\ = \sum_{d=1}^N U_{rq}^{(2,2)}(b, g, f, a) U_{p'pq}^{(2)}(g, d, e, f) U_{p'pr}^{(2)}(b, c, d, g), \end{aligned} \quad (8)$$

which is equation (17) of Baxter [10]. The product of  $L$  such squares,  $\mathbf{U}(t_q)$ , has trace  $\tau_2(t_q)$  when the cyclic boundary condition  $\sigma_{L+1} = \sigma_1$  and  $\sigma'_{L+1} = \sigma'_1$  is imposed, i.e.

$$\tau_2(t_q) = \prod_{J=1}^L U_{p'pq}^{(2)}(\sigma_J, \sigma_{J+1}, \sigma'_{J+1}, \sigma'_J) = \text{tr } \mathbf{U}(t_q). \quad (9)$$

Following common practice we write

$$\mathbf{U}(t_q) = \begin{bmatrix} \mathbf{A}(t_q) & \mathbf{B}(t_q) \\ \mathbf{C}(t_q) & \mathbf{D}(t_q) \end{bmatrix} = \sum_{j=0}^L (-\omega t)^j \begin{bmatrix} \mathbf{A}_j & \mathbf{B}_j \\ \mathbf{C}_j & \mathbf{D}_j \end{bmatrix}, \quad t = t_q/c_{pp'}, \quad (10)$$

where  $c_{pp'}$  is some constant. This satisfies a Yang–Baxter equation like (8). Since the  $U_{rq}^{(2,2)}$  are the weights of a six-vertex model,  $\mathbf{U}(t_q)$  is a cyclic representation of quantum group  $U_q(\mathfrak{sl}_2)$  [12]. This structure is intimately related to that on the XXZ model [9, 13].

From the Yang–Baxter equation (8), we find sixteen relations between the four elements of  $\mathbf{U}(t)$  in (10), of which four of them are listed here

$$(1 - \omega^{-1}x/y)\mathbf{A}(x)\mathbf{B}(y) = (1 - x/y)\mathbf{B}(y)\mathbf{A}(x) + (1 - \omega^{-1})\mathbf{A}(y)\mathbf{B}(x), \quad (11)$$

$$(1 - \omega^{-1}x/y)\mathbf{A}(y)\mathbf{C}(x) = \omega^{-1}(1 - x/y)\mathbf{C}(x)\mathbf{A}(y) + (1 - \omega^{-1})\mathbf{A}(x)\mathbf{C}(y), \quad (12)$$

$$(1 - \omega^{-1}x/y)\mathbf{C}(x)\mathbf{D}(y) = (1 - x/y)\mathbf{D}(y)\mathbf{C}(x) + (1 - \omega^{-1})\mathbf{C}(y)\mathbf{D}(x), \quad (13)$$

$$(1 - \omega^{-1}x/y)\mathbf{B}(y)\mathbf{D}(x) = \omega^{-1}(1 - x/y)\mathbf{D}(x)\mathbf{B}(y) + (1 - \omega^{-1})\mathbf{B}(x)\mathbf{D}(y), \quad (14)$$

where  $x = t_q$ , and  $y = t_r$ .

### 1.2. Superintegrable $\tau_2(t_q)$

Now we restrict ourselves to the superintegrable case with  $\{x_{p'}, y_{p'}, \mu_{p'}\} = \{y_p, x_p, 1/\mu_p\}$ . After dropping the subscripts and the factors  $(\mu_p/y_p)^{\alpha-\beta}$ , which can be done only for the homogeneous case, the nonvanishing squares in (4) are

$$\begin{aligned} U^{(2)}(a, b, b, a) &= 1 - \omega^{a-b+1}t \rightarrow \mathbf{1} - \omega t \mathbf{Z}, \\ U^{(2)}(a, b, b-1, a) &= -\omega t(1 - \omega^{a-b+1}) \rightarrow -\omega t(\mathbf{1} - \mathbf{Z})\mathbf{X} = -\omega t(1 - \omega)\mathbf{f}, \\ U^{(2)}(a, b, b, a-1) &= (1 - \omega^{a-b}) \rightarrow \mathbf{X}^{-1}(\mathbf{1} - \mathbf{Z}) = (1 - \omega)\mathbf{e}, \\ U^{(2)}(a, b, b-1, a-1) &= \omega(\omega^{a-b} - t) \rightarrow (\omega \mathbf{Z} - \omega t \mathbf{1}), \end{aligned} \quad (15)$$

where  $t = t_q/t_p$ , or  $c_{pp'} = t_p$  in (10). As these squares are functions of the differences of the pairs of adjacent spins, defined in [6] as the edge variables  $n = a - b$ , the operators acting on the edge variables are

$$\mathbf{Z}_{m,n} = \delta_{m,n} \omega^n, \quad \mathbf{Z}|n\rangle = \omega^n|n\rangle, \quad \mathbf{X}_{m,n} = \delta_{m,n+1}, \quad \mathbf{X}|n\rangle = |n+1\rangle, \quad n = a - b. \quad (16)$$

which can be extended to  $L$  edges  $n_j = \sigma_j - \sigma_{j+1}$  for  $j = 1, \dots, L$ , as

$$\mathbf{X}_j = \overset{1}{\mathbf{1}} \otimes \cdots \otimes \overset{j}{\mathbf{1}} \otimes \mathbf{X} \otimes \overset{\dots}{\mathbf{1}} \otimes \cdots \otimes \overset{L}{\mathbf{1}}, \quad \mathbf{Z}_j = \overset{1}{\mathbf{1}} \otimes \cdots \otimes \overset{j}{\mathbf{1}} \otimes \mathbf{Z} \otimes \overset{\dots}{\mathbf{1}} \otimes \cdots \otimes \overset{L}{\mathbf{1}}. \quad (17)$$

The periodic boundary condition is equivalent to  $n_1 + \dots + n_L \equiv 0 \pmod{N}$ ; thus there are only  $N^{L-1}$  edge variables. As the products of the squares  $\mathbf{U}$  in (10) are functions of the edges variables only, the transfer matrix  $\tau_2(t_q)$  being the trace of the  $N^L$  spin variables is block cyclic; each blocks is of the size  $N^{L-1} \times N^{L-1}$ , and becomes block-diagonal after Fourier transform, with  $N$  diagonal blocks

$$\tau_2(t_q)|_Q = \mathbf{A}(t_q) + \omega^{-Q} \mathbf{D}(t_q), \quad Q = 0, \dots, N-1. \quad (18)$$

The leading coefficients in (10) are easily found, see (I.24) and (I.25)<sup>‡</sup>,

$$\mathbf{A}_0 = \mathbf{D}_L = \mathbf{1}, \quad \mathbf{A}_L = \mathbf{D}_0 \omega^{-L} = \prod_{j=1}^L \mathbf{Z}_j, \quad \mathbf{C}_L = \mathbf{B}_0 = 0, \quad (19)$$

$$\begin{aligned} \mathbf{B}_L &= (1 - \omega) \sum_{j=1}^L \prod_{m=1}^{j-1} \mathbf{Z}_m \mathbf{f}_j, & \mathbf{C}_0 &= (1 - \omega) \sum_{j=1}^L \omega^{j-1} \prod_{m=1}^{j-1} \mathbf{Z}_m \mathbf{e}_j, \\ \mathbf{B}_1 &= (1 - \omega) \sum_{j=1}^L \omega^{L-j} \mathbf{f}_j \prod_{m=j+1}^L \mathbf{Z}_m, & \mathbf{C}_{L-1} &= (1 - \omega) \sum_{j=1}^L \mathbf{e}_j \prod_{m=j+1}^L \mathbf{Z}_m. \end{aligned} \quad (20)$$

### 1.3. Relationship with generators of $U_q(\mathfrak{sl}_2)$

The generators  $\mathbf{e}_j$  and  $\mathbf{f}_j$  in the above equations are defined by

$$(1 - \omega) \mathbf{e}_j = \mathbf{X}_j^{-1} (\mathbf{1} - \mathbf{Z}_j), \quad (1 - \omega) \mathbf{f}_j = (\mathbf{1} - \mathbf{Z}_j) \mathbf{X}_j, \quad (21)$$

and satisfy the relation

$$(1 - \omega)(\mathbf{e}_j \mathbf{f}_j - \omega \mathbf{f}_j \mathbf{e}_j) = (\mathbf{1} - \omega \mathbf{Z}_j^2). \quad (22)$$

They are not the same as the usual  $\mathbf{e}'_j$  and  $\mathbf{f}'_j$  of the quantum group  $U_q(\mathfrak{sl}_2)$ , but are related by

$$\mathbf{e}'_j = -q \mathbf{e}_j \mathbf{Z}_j^{-1/2}, \quad \mathbf{f}'_j = q \mathbf{Z}_j^{-1/2} \mathbf{f}_j, \quad \omega = q^2, \quad (23)$$

as shown in [14]. These satisfy the relation

$$(q - q^{-1})(\mathbf{e}'_j \mathbf{f}'_j - \mathbf{f}'_j \mathbf{e}'_j) = q \mathbf{Z}_j - (q \mathbf{Z}_j)^{-1}, \quad (24)$$

as defined by Jimbo [12]. This difference in these operators is due to the fact that the six-vertex model in (6) are not symmetric.

<sup>‡</sup> All equations in [6] are denoted here by prefacing I to the equation number, those in [7] by prefacing II, and those in [8] by adding III.

## 1.4. Commutation relations

We use (11) to (14) to derive commutation relations. Equating the coefficients of  $x^{L+1}$  in (11) and the coefficients of  $x^0$  in (14) where  $\mathbf{B}_0 = 0$ , we find

$$\mathbf{A}_L \mathbf{B}(y) = \omega \mathbf{B}(y) \mathbf{A}_L, \quad \mathbf{D}_0 \mathbf{B}(y) = \omega \mathbf{B}(y) \mathbf{D}_0. \quad (25)$$

In the limit  $y \rightarrow 0$ , we have  $\mathbf{B}(y) \rightarrow -\omega y \mathbf{B}_1$  as  $\mathbf{B}_0 = 0$ , so that (11) becomes

$$\mathbf{A}(x) \mathbf{B}_1 - \omega \mathbf{B}_1 \mathbf{A}(x) = (1 - \omega^{-1}) x^{-1} \mathbf{A}_0 \mathbf{B}(x) = (1 - \omega^{-1}) x^{-1} \mathbf{B}(x), \quad (26)$$

using  $\mathbf{A}_0 = \mathbf{1}$ . By equating the coefficients of  $y^L$  in (11), we find

$$\mathbf{A}(x) \mathbf{B}_L - \mathbf{B}_L \mathbf{A}(x) = (1 - \omega^{-1}) \mathbf{A}_L \mathbf{B}(x) = (\omega - 1) \mathbf{B}(x) \mathbf{A}_L, \quad (27)$$

where (25) has been used. Similarly, equating the coefficients of  $y^L$  in (12) and of  $y^{-1}$  in (13), and also the coefficients of  $x^0$  and  $x^L$  in (12), we find

$$\begin{aligned} \mathbf{A}_L \mathbf{C}(x) &= \omega^{-1} \mathbf{C}(x) \mathbf{A}_L, & \mathbf{D}_0 \mathbf{C}(x) &= \omega^{-1} \mathbf{C}(x) \mathbf{D}_0, \\ \mathbf{A}(y) \mathbf{C}_0 - \omega^{-1} \mathbf{C}_0 \mathbf{A}(y) &= (1 - \omega^{-1}) \mathbf{C}(y), \\ \mathbf{A}(y) \mathbf{C}_{L-1} - \mathbf{C}_{L-1} \mathbf{A}(y) &= (\omega - 1) y \mathbf{C}(y) \mathbf{A}_L, \end{aligned} \quad (28)$$

using  $\mathbf{C}_L = 0$  and  $\mathbf{A}_0 = \mathbf{1}$ . In the same way, (13) and (14) yield the relations

$$\begin{aligned} \mathbf{D}(y) \mathbf{C}_0 - \mathbf{C}_0 \mathbf{D}(y) &= -(1 - \omega^{-1}) \mathbf{C}(y) \mathbf{D}_0, \\ \mathbf{D}(y) \mathbf{C}_{L-1} - \omega^{-1} \mathbf{C}_{L-1} \mathbf{D}(x) &= -(\omega - 1) y \mathbf{C}(y), \\ \mathbf{D}(x) \mathbf{B}_1 - \mathbf{B}_1 \mathbf{D}(x) &= -(1 - \omega^{-1}) x^{-1} \mathbf{B}(x) \mathbf{D}_0, \\ \mathbf{D}(x) \mathbf{B}_L - \omega \mathbf{B}_L \mathbf{D}(x) &= -(\omega - 1) \mathbf{B}(x). \end{aligned} \quad (29)$$

It is straightforward to prove by induction the following relations,

$$\mathbf{A}(x) \mathbf{C}_0^n = \omega^{-n} \mathbf{C}_0^n \mathbf{A}(x) + (\omega - 1) \omega^{-n} [n] \mathbf{C}_0^{n-1} \mathbf{C}(x), \quad (30)$$

$$\mathbf{D}(x) \mathbf{C}_0^n = \mathbf{C}_0^n \mathbf{D}(x) - (\omega - 1) \omega^{-n} [n] \mathbf{C}_0^{n-1} \mathbf{C}(x) \mathbf{D}_0, \quad (31)$$

$$\mathbf{A}(x) \mathbf{B}_1^n = \omega^n \mathbf{B}_1^n \mathbf{A}(x) + (1 - \omega^{-1}) x^{-1} [n] \mathbf{B}_1^{n-1} \mathbf{B}(x), \quad (32)$$

$$\mathbf{D}(x) \mathbf{B}_1^n = \mathbf{B}_1^n \mathbf{D}(x) - (1 - \omega^{-1}) x^{-1} [n] \mathbf{B}_1^{n-1} \mathbf{B}(x) \mathbf{D}_0, \quad (33)$$

where  $[n] \equiv 1 + \dots + \omega^{n-1}$ . Similar relations for  $\mathbf{B}_L$  and  $\mathbf{C}_{L-1}$  are

$$\mathbf{A}(x) \mathbf{C}_{L-1}^n = \mathbf{C}_{L-1}^n \mathbf{A}(x) + (\omega - 1) \omega^{1-n} x [n] \mathbf{C}_{L-1}^{n-1} \mathbf{C}(x) \mathbf{A}_L, \quad (34)$$

$$\mathbf{D}(x) \mathbf{C}_{L-1}^n = \omega^{-n} \mathbf{C}_{L-1}^n \mathbf{D}(x) - (\omega - 1) \omega^{1-n} x [n] \mathbf{C}_{L-1}^{n-1} \mathbf{C}(x), \quad (35)$$

$$\mathbf{A}(x) \mathbf{B}_L^n = \mathbf{B}_L^n \mathbf{A}(x) + (\omega - 1) [n] \mathbf{B}_L^{n-1} \mathbf{B}(x) \mathbf{A}_L, \quad (36)$$

$$\mathbf{D}(x) \mathbf{B}_L^n = \omega^n \mathbf{B}_L^n \mathbf{D}(x) - (\omega - 1) [n] \mathbf{B}_L^{n-1} \mathbf{B}(x). \quad (37)$$

2. Eigenvectors of  $\tau_2(t_q)|_Q$ 

We shall find the eigenvectors  $\nu_Q$  of  $\tau_2(t_q)|_Q$  such that

$$\tau_2(t_q)|_Q \nu_Q = [(1 - \omega t)^L + \omega^{-Q} (1 - t)^L] \nu_Q, \quad \text{or} \quad (38)$$

$$\tau_2(t_q)|_Q \nu_Q = [\omega^{-Q} (1 - \omega t)^L + (1 - t)^L] \nu_Q, \quad (39)$$

where  $t = t_q/t_p$ . Defining similarly as in [14]

$$B_j^{(n)} = \lim_{\substack{q \rightarrow \omega \\ |q| < 1}} \frac{B_j^n}{[n]!}, \quad \text{with } [n] = \frac{1 - q^n}{1 - q}, \quad [n]! = [n] \cdots [2] [1], \quad (40)$$

and using (30) and (32), we can show

$$\begin{aligned} \mathbf{A}(x) \mathbf{C}_0^{(nN+Q)} \mathbf{B}_1^{(mN+Q)} &= \omega^{-Q} [\mathbf{C}_0^{(nN+Q)} \mathbf{A}(x) + (\omega - 1) \mathbf{C}_0^{(nN+Q-1)} \mathbf{C}(x)] \mathbf{B}_1^{(mN+Q)} \\ &= \omega^{-Q} \mathbf{C}_0^{(nN+Q)} [\omega^Q \mathbf{B}_1^{(mN+Q)} \mathbf{A}(x) + (1 - \omega^{-1}) x^{-1} \mathbf{B}_1^{(mN+Q-1)} \mathbf{B}(x)] \\ &\quad + \omega^{-Q} (\omega - 1) \mathbf{C}_0^{(nN+Q-1)} \mathbf{C}(x) \mathbf{B}_1^{(mN+Q)}, \end{aligned} \quad (41)$$

while (31) and (33) yield

$$\begin{aligned} \mathbf{D}(x) \mathbf{C}_0^{(nN+Q)} \mathbf{B}_1^{(mN+Q)} &= \mathbf{C}_0^{(nN+Q)} [\mathbf{B}_1^{(mN+Q)} \mathbf{D}(x) - (1 - \omega^{-1}) x^{-1} \mathbf{B}_1^{(mN+Q-1)} \mathbf{B}(x)] \mathbf{D}_0 \\ &\quad - (\omega - 1) \mathbf{C}_0^{(nN+Q-1)} \mathbf{C}(x) \mathbf{B}_1^{(mN+Q)} \mathbf{D}_0. \end{aligned} \quad (42)$$

Consequently, we find that

$$\begin{aligned} [\mathbf{A}(x) + \omega^{-Q} \mathbf{D}(x), \mathbf{C}_0^{(nN+Q)} \mathbf{B}_1^{(mN+Q)}] &= \omega^{-Q} (\omega - 1) [(\omega x)^{-1} \mathbf{C}_0^{(nN+Q)} \mathbf{B}_1^{(mN+Q-1)} \mathbf{B}(x) \\ &\quad + \mathbf{C}_0^{(nN+Q-1)} \mathbf{C}(x) \mathbf{B}_1^{(mN+Q)}] [1 - \mathbf{D}_0]. \end{aligned} \quad (43)$$

As seen from (19), we have  $\mathbf{D}_0 = \omega^L \prod_{j=1}^L \mathbf{Z}_j$ , so that for  $L$  a multiple of  $N$ , and for  $|\{n_j\}\rangle$  with  $n_1 + \cdots + n_L \equiv 0 \pmod{N}$ , we have  $(1 - \mathbf{D}_0)|\{n_j\}\rangle = 0$ . Hence,

$$[\mathbf{A}(x) + \omega^{-Q} \mathbf{D}(x), \mathbf{C}_0^{(nN+Q)} \mathbf{B}_1^{(mN+Q)}] |\{n_j\}\rangle = 0. \quad (44)$$

Similarly, we may prove

$$\begin{aligned} [\mathbf{A}(x) + \omega^Q \mathbf{D}(x), \mathbf{B}_1^{(mN+Q)} \mathbf{C}_0^{(nN+Q)}] |\{n_j\}\rangle &= 0, \\ [\mathbf{A}(x) + \omega^{-Q} \mathbf{D}(x), \mathbf{B}_L^{(mN+Q)} \mathbf{C}_{L-1}^{(nN+Q)}] |\{n_j\}\rangle &= 0, \\ [\mathbf{A}(x) + \omega^Q \mathbf{D}(x), \mathbf{C}_{L-1}^{(nN+Q)} \mathbf{B}_L^{(mN+Q)}] |\{n_j\}\rangle &= 0. \end{aligned} \quad (45)$$

Particularly, the ferromagnetic ground state  $|\Omega\rangle = |\{0\}\rangle$  and the antiferromagnetic ground state  $|\bar{\Omega}\rangle = |\{N-1\}\rangle$  are easily seen, from either (15) or (I.29), to satisfy

$$\tau_2(t_q)|_Q |\Omega\rangle = [(1 - \omega t)^L + \omega^{-Q} (1 - t)^L] |\Omega\rangle, \quad (46)$$

$$\tau_2(t_q)|_Q |\bar{\Omega}\rangle = [\omega^{-Q} (1 - \omega t)^L + (1 - t)^L] |\bar{\Omega}\rangle. \quad (47)$$

Due to (44) and (45), we find that

$$\prod_{j=1}^J \mathbf{C}_0^{(m_j N+Q)} \mathbf{B}_1^{(n_j N+Q)} |\Omega\rangle, \quad \prod_{j=1}^J \mathbf{C}_{L-1}^{(m_j N+N-Q)} \mathbf{B}_L^{(n_j N+N-Q)} |\Omega\rangle \quad (48)$$

are eigenvectors in the same degenerate eigenspace as  $|\Omega\rangle$ , while

$$\prod_{j=1}^J \mathbf{B}_1^{(m_j N+N-Q)} \mathbf{C}_0^{(n_j N+N-Q)} |\bar{\Omega}\rangle, \quad \prod_{j=1}^J \mathbf{B}_L^{(m_j N+Q)} \mathbf{C}_{L-1}^{(n_j N+Q)} |\bar{\Omega}\rangle \quad (49)$$

are eigenvectors in the same degenerate eigenspace as  $|\bar{\Omega}\rangle$ . For  $Q \neq 0$ , we conclude from calculations for  $N, L$  small, that these two eigenspaces have dimension  $2^{r-1}$  ( $m_Q = r-1$ ). Thus by letting  $0 \leq m_1 < n_1 < \dots < m_J < n_J \leq r-1$ , where  $0 \leq J \leq r-1$  and  $\sum_{j=1}^J (n_j - m_j) = J$ , similar to the results given in [15], we can obtain a basis of  $2^{r-1}$  eigenvectors in each of the two eigenspaces corresponding to the two eigenvalues. For  $Q = 0$ , it is easily seen from (46) and (47) that the two eigenvalues become equal and the two eigenspaces merge into one.

From (I.47) in [6], we find that the eigenvectors of the degenerate eigenspaces can also be obtained differently. It is far from obvious how to find the generators of the loop algebra. We now use the information obtained in the  $Q = 0$  case to find maybe the best choices in the  $Q \neq 0$  cases.

### 3. Quantum Loop Subalgebra

We now shall present the generators of the  $\mathfrak{sl}_2$  algebras and generators of the loop algebra.

#### 3.1. Drinfeld polynomials

From (20), we find that

$$\begin{aligned} \bar{\mathbf{C}}_0^{(m)} &= \mathbf{C}_0^{(m)} (1 - \omega)^{-m} = \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = m}} \prod_{j=1}^L \mathbf{z}_j^{\bar{N}_j} \frac{\omega^{(j-1)n_j} \mathbf{e}_j^{n_j}}{[n_j]!}, \quad \bar{N}_j = \sum_{\ell=j+1}^L n_\ell, \\ \bar{\mathbf{B}}_1^{(m)} &= \mathbf{B}_1^{(m)} (1 - \omega)^{-m} = \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = m}} \prod_{j=1}^L \frac{\omega^{-jn_j} \mathbf{f}_j^{n_j}}{[n_j]!} \mathbf{z}_j^{N_j}, \quad N_j = \sum_{\ell=1}^{j-1} n_\ell. \end{aligned} \quad (50)$$

Using (21) or (II.52), we find

$$\bar{\mathbf{C}}_0^{(mN+Q)} \bar{\mathbf{B}}_1^{(mN+Q)} |\Omega\rangle = \omega^{-Q} \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = mN+Q}} |\Omega\rangle = \omega^{-Q} \Lambda_m^Q |\Omega\rangle. \quad (51)$$

Here the  $\Lambda_m^Q$  are the coefficients of the Drinfeld polynomial  $P_Q(z)$  in (1). However, (20) also yields

$$\bar{\mathbf{C}}_{L-1}^{(m)} = \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = m}} \prod_{j=1}^L \mathbf{z}_j^{N_j} \frac{\mathbf{e}_j^{n_j}}{[n_j]!}, \quad \bar{\mathbf{B}}_L^{(m)} = \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = m}} \prod_{j=1}^L \frac{\mathbf{f}_j^{n_j}}{[n_j]!} \mathbf{z}_j^{\bar{N}_j}, \quad (52)$$

so that

$$\bar{\mathbf{C}}_{L-1}^{(mN+N-Q)} \bar{\mathbf{B}}_L^{(mN+N-Q)} |\Omega\rangle = \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = mN+N-Q}} |\Omega\rangle = \Lambda_m^{N-Q} |\Omega\rangle. \quad (53)$$

Now the  $\Lambda_m^{N-Q}$  are the coefficients of the polynomial  $P_{N-Q}(z)$ , whose roots are the inverses of the roots of  $P_Q(z)$ . We have the situation that the two sets of eigenvectors in (48) have the same eigenvalues, but correspond to different Drinfeld polynomials. On

the other hand, the coefficients of the Drinfeld polynomial are symmetric  $\Lambda_m = \Lambda_{r-m}$  for  $Q = 0$ , so that roots of the polynomial then appear in pairs  $z_j$  and  $1/z_j$ . Since the algebra and the roots of the Drinfeld Polynomials are intimately related [6, 7, 17], the corresponding algebras for  $Q \neq 0$  cases are different from the algebra for the  $Q = 0$  case. We shall explore this in more detail.

### 3.2. Generators $\mathbf{E}_{m,Q}^\pm$ on the ground state

In (1), we have let  $z_{m,Q}$  denote the roots of the Drinfeld polynomial  $P_Q(z)$ , and we again define the polynomial, [see (II.11) or (III.53) [8, 16]]

$$f_j^Q(z) = \prod_{\ell \neq j} \frac{z - z_{\ell,Q}}{z_{j,Q} - z_{\ell,Q}} = \sum_{n=0}^{m_Q-1} \beta_{j,n}^Q z^n, \quad f_j^Q(z_{k,Q}) = \delta_{j,k}, \quad (54)$$

where  $\beta_{j,n}^Q$  are the elements of the inverse of the Vandermonde matrix, such that

$$\sum_{n=0}^{m_Q-1} \beta_{j,n}^Q z_{k,Q}^n = \delta_{j,k}, \quad \sum_{k=1}^{m_Q} z_{k,Q}^n \beta_{k,m}^Q = \delta_{n,m}, \quad \text{for } 0 \leq n \leq m_Q - 1. \quad (55)$$

Thus we generalize the previous results to include the cases for  $Q \neq 0$ . We may also generalize (II.53) and (II.54) to

$$\langle \Omega | \mathbf{E}_{m,Q}^- = -\omega^Q (\beta_{m,0}^Q / \Lambda_0^Q) \sum_{\ell=1}^{m_Q} z_{\ell,Q}^{\ell-1} \langle \Omega | \bar{\mathbf{C}}_0^{(\ell N+Q)} \bar{\mathbf{B}}_1^{(\ell N-N+Q)}, \quad (56)$$

$$\mathbf{E}_{m,Q}^+ |\Omega\rangle = \omega^Q (\beta_{m,0}^Q / \Lambda_0^Q) \sum_{\ell=1}^{m_Q} z_{\ell,Q}^\ell \bar{\mathbf{C}}_0^{(\ell N-N+Q)} \bar{\mathbf{B}}_1^{(\ell N+Q)} |\Omega\rangle. \quad (57)$$

If the  $\mathbf{E}_{m,Q}^\pm$  are generators of  $\mathfrak{sl}_2$  algebras, then it is necessary that

$$\langle \Omega | \mathbf{E}_{k,Q}^- \mathbf{E}_{m,Q}^+ | \Omega \rangle = -\delta_{k,m} \langle \Omega | \mathbf{H}_k^Q | \Omega \rangle = \delta_{k,m}. \quad (58)$$

To show this, we use (50) and (II.55) to obtain

$$\langle \Omega | \bar{\mathbf{C}}_0^{(\ell N+N+Q)} \bar{\mathbf{B}}_1^{(\ell N+Q)} = \omega^{-Q} \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \langle \{n_j\} | \omega^{\sum_j j n_j} \bar{K}_{\ell N+Q}(\{n_j\}), \quad (59)$$

$$\bar{\mathbf{C}}_0^{(\ell N+Q)} \bar{\mathbf{B}}_1^{(\ell N+N+Q)} |\Omega\rangle = \omega^{-Q} \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \omega^{-\sum_j j n_j} K_{\ell N+Q}(\{n_j\}) |\{n_j\}\rangle, \quad (60)$$

where  $K_m(\{n_j\})$  and  $\bar{K}_m(\{n_j\})$  are defined in (III.4) and (III.5). These equations are similar to (II.59). Substituting the above equations into (56) and (57), then using (II.63) and (II.64) (or (III.13)), we find

$$\langle \Omega | \mathbf{E}_{m,Q}^- = -(\beta_{m,0}^Q / \Lambda_0^Q) \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \langle \{n_j\} | \omega^{\sum_j j n_j} \bar{G}_Q(\{n_j\}, z_{m,Q}), \quad (61)$$

$$\mathbf{E}_{k,Q}^+ |\Omega\rangle = (\beta_{k,0}^Q / \Lambda_0^Q) z_{k,Q} \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \omega^{-\sum_j j n_j} G_Q(\{n_j\}, z_{k,Q}) |\{n_j\}\rangle. \quad (62)$$



We now use the theorem in [8] to prove (58). From (54), we find

$$\beta_{m,0}^Q = \prod_{\ell \neq m} \frac{-z_{\ell,Q}}{z_{m,Q} - z_{\ell,Q}} = -\frac{\Lambda_0^Q}{\Lambda_{m,Q}^Q z_{m,Q}} \prod_{\ell \neq m} \frac{1}{z_{m,Q} - z_{\ell,Q}}, \quad (63)$$

so that the constant in (III.16) becomes

$$B_{m,Q} = (\Lambda_{m,Q}^Q)^2 z_{m,Q} \prod_{\ell \neq m} (z_{m,Q} - z_{\ell,Q})^2 = (\Lambda_0^Q / \beta_{m,0}^Q)^2 z_{m,Q}^{-1}, \quad (64)$$

Consequently, we may combine (61) and (62), and then use (III.15) to get

$$\langle \Omega | \mathbf{E}_{k,Q}^- \mathbf{E}_{m,Q}^+ | \Omega \rangle = -\frac{z_{k,Q} \beta_{k,0}^Q \beta_{m,0}^Q}{(\Lambda_0^Q)^2} \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \bar{G}_Q(\{n_j\}, z_{m,Q}) G_Q(\{n_j\}, z_{k,Q}) = \delta_{k,m}. \quad (65)$$

This is the first evidence that the above generalization of (II.53) and (II.54) to  $Q \neq 0$  cases is correct.

### 3.3. Generators $\mathbf{x}_{m,Q}^\pm$ on the ground state

In paper [7], we have studied the  $Q = 0$  case, for which the generators  $\mathbf{x}_m^\pm$  of the loop algebra are known. From these operators, we obtained the  $\mathbf{E}_{m,Q}^\pm$ , the generators of  $\mathfrak{sl}_2$ 's. In this paper, we will go in the reverse order, by using (56) and (57) to determine the best form of  $\mathbf{x}_{m,Q}^\pm$ . As in (II.50) we let

$$S_n^Q = \sum_{m=1}^{m_Q} \beta_{m,0}^Q z_{m,Q}^{-n}, \quad S_n^Q = 0, \quad \text{for } 1 - m_Q \leq n < 0, \quad (66)$$

where (55) is used to show  $S_n^Q = 0$  for  $1 - m_Q \leq n < 0$ . Now similar to (II.12), we let

$$\begin{aligned} \mathbf{x}_{n,Q}^- | \Omega \rangle &= \sum_{m=1}^{m_Q} z_{m,Q}^{-n} \mathbf{E}_{m,Q}^+ | \Omega \rangle = \omega^Q \sum_{m=1}^{m_Q} z_{m,Q}^{-n} (\beta_{m,0}^Q / \Lambda_0^Q) \sum_{\ell=1}^{m_Q} z_{m,Q}^\ell \bar{\mathbf{C}}_0^{(\ell N - N + Q)} \bar{\mathbf{B}}_1^{(\ell N + Q)} | \Omega \rangle \\ &= (\omega^Q / \Lambda_0^Q) \sum_{\ell=1}^n S_{n-\ell}^Q \bar{\mathbf{C}}_0^{(\ell N - N + Q)} \bar{\mathbf{B}}_1^{(\ell N + Q)} | \Omega \rangle, \end{aligned} \quad (67)$$

where (57) and (66) are used. Similarly we find from (56)

$$\langle \Omega | \mathbf{x}_{n,Q}^+ = - \sum_{m=1}^{m_Q} z_{m,Q}^{-n} \langle \Omega | \mathbf{E}_{m,Q}^- = (\omega^Q / \Lambda_0^Q) \sum_{\ell=0}^n S_{n-\ell}^Q \langle \Omega | \bar{\mathbf{C}}_0^{(\ell N + N + Q)} \bar{\mathbf{B}}_1^{(\ell N + Q)}. \quad (68)$$

These are generalizations of (II.45) and (II.46).

Furthermore, the relation (II.44) can be generalized to

$$\sum_{n=0}^m \Lambda_{m-n}^Q S_n^Q = \Lambda_0^Q \delta_{m,0} \quad \text{with } S_0^Q = 1. \quad (69)$$

To show this, we can see that for  $m = 0$  we have  $S_0^Q = 1$ , while for  $m \geq 1$  we write

$$\sum_{n=0}^m \Lambda_{m-n}^Q S_n^Q = \sum_{n=0}^m \Lambda_n^Q S_{m-n}^Q = \sum_{n=0}^{m_Q} \Lambda_n^Q \sum_{\ell=1}^{m_Q} \beta_{\ell,0}^Q z_{\ell,Q}^{n-m} = \sum_{\ell=1}^{m_Q} \beta_{\ell,0}^Q z_{\ell,Q}^{-m} \sum_{n=0}^{m_Q} \Lambda_n^Q z_{\ell,Q}^n = 0. \quad (70)$$

where the summation over  $n$  has been extended to  $0 \leq n \leq m_Q$  as  $S_{m-n}^Q = 0$  for  $m < n \leq m_Q$ . Next, we have substituted (66) into the sum and interchanged the order of summation to find it is identically zero for  $m \neq 0$ , as  $z_{\ell,Q}$  are the roots of the Drinfeld polynomial. Thus (69) holds.

Using (67) and (69) we generalize (I.42) to

$$\begin{aligned} \sum_{n=1}^m \Lambda_{m-n}^Q \mathbf{x}_{n,Q}^- |\Omega\rangle &= (\omega^Q / \Lambda_0^Q) \sum_{\ell=1}^m \sum_{n=\ell}^m \Lambda_{m-n}^Q S_{n-\ell}^Q \bar{\mathbf{C}}_0^{(\ell N - N + Q)} \bar{\mathbf{B}}_1^{(\ell N + Q)} |\Omega\rangle \\ &= \omega^Q \sum_{\ell=1}^m \delta_{m,\ell} \bar{\mathbf{C}}_0^{(\ell N - N + Q)} \bar{\mathbf{B}}_1^{(\ell N + Q)} |\Omega\rangle = \omega^Q \bar{\mathbf{C}}_0^{(m N - N + Q)} \bar{\mathbf{B}}_1^{(m N + Q)} |\Omega\rangle. \end{aligned} \quad (71)$$

For  $m > m_Q = \lfloor (L(N-1) + Q)/N \rfloor$ , the right hand side is identically zero, so that there are  $m_Q$  independent vectors  $\mathbf{x}_{n,Q}^- |\Omega\rangle$ . Similarly, from (68) and (69) we may derive

$$\sum_{n=0}^m \Lambda_{m-n}^Q \langle \Omega | \mathbf{x}_{n,Q}^+ = \omega^Q \langle \Omega | \bar{\mathbf{C}}_0^{(m N + N + Q)} \bar{\mathbf{B}}_1^{(m N + Q)}. \quad (72)$$

### 3.4. Generators $\mathbf{h}_{m,Q}$ on the ground state

We define

$$d_{m,Q} = \langle \Omega | \mathbf{h}_{m,Q} | \Omega \rangle = \langle \Omega | \mathbf{x}_{m-1,Q}^+ \mathbf{x}_{1,Q}^- | \Omega \rangle, \quad \text{for } 1 \leq n < \infty. \quad (73)$$

Substituting (67) and (68) into the above equation and using (59) and (60), we find

$$d_{m,Q} = (\Lambda_0^Q)^{-2} \sum_{\ell=0}^{m-1} S_{m-1-\ell}^Q \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \bar{K}_{\ell N + Q}(\{n_j\}) K_Q(\{n_j\}). \quad (74)$$

First we use Lemma 2(i) [8] or (III.33), followed by (66) also extending the interval of summation to  $1 \leq \ell \leq m_Q$ , and lastly we use the identities (III.52) and (63), to obtain

$$d_{m,Q} = (\Lambda_0^Q)^{-1} \sum_{\ell=1}^m S_{m-\ell}^Q \ell \Lambda_\ell^Q \quad (75)$$

$$= (\Lambda_0^Q)^{-1} \sum_{j=1}^{m_Q} \beta_{j,0}^Q z_{j,Q}^{1-m} \sum_{\ell=1}^{m_Q} \ell \Lambda_\ell^Q z_{j,Q}^{\ell-1} = - \sum_{j=1}^{m_Q} z_{j,Q}^{-m}. \quad (76)$$

This then generalizes (II.A.3). Using (73) and (72) and (III.33) of Lemma 2, we find

$$\begin{aligned} \sum_{n=1}^m \Lambda_{m-n}^Q d_{n,Q} &= \sum_{n=0}^{m-1} \Lambda_{m-1-n}^Q \langle \Omega | \mathbf{x}_{n,Q}^+ \mathbf{x}_{1,Q}^- | \Omega \rangle \\ &= \frac{\omega^Q}{\Lambda_0^Q} \langle \Omega | \bar{\mathbf{C}}_0^{(m N + Q)} \bar{\mathbf{B}}_1^{(m N - N + Q)} \bar{\mathbf{C}}_0^{(Q)} \bar{\mathbf{B}}_1^{(N + Q)} | \Omega \rangle \\ &= (\Lambda_0^Q)^{-1} \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \bar{K}_{m N - N + Q}(\{n_j\}) K_Q(\{n_j\}) \\ &= m \Lambda_m^Q, \end{aligned} \quad (77)$$

which is relation in (II.A.1). Next, we shall show

$$d_{m,Q} = \langle \Omega | \mathbf{h}_{m,Q} | \Omega \rangle = \langle \Omega | \mathbf{x}_{m-k,Q}^+ \mathbf{x}_{k,Q}^- | \Omega \rangle, \quad \text{for } 1 < k \leq m, \quad (78)$$

which is the necessary condition that the loop algebra or subalgebra exists. Again we substitute (67) and (68) into the right hand side of the above equation, then use (59) and (60), and finally use (III.34) of Lemma 2 in [8], to find

$$\begin{aligned} \langle \Omega | \mathbf{x}_{m-k,Q}^+ \mathbf{x}_{k,Q}^- | \Omega \rangle &= (\Lambda_0^Q)^{-2} \sum_{\ell=0}^{m-k} S_{m-k-\ell}^Q \sum_{n=0}^{k-1} S_{k-1-n}^Q \\ &\quad \times \sum_{j=0}^{\ell} (n - \ell + 1 + 2j) \Lambda_{\ell-j}^Q \Lambda_{n+1+j}^Q. \end{aligned} \quad (79)$$

Interchanging the order of summation over  $\ell$  with the one over  $j$  and then letting  $\ell' = \ell - j$ , we find that the summation over  $\ell'$  can be carried out by using (69) and (75). We obtain

$$\begin{aligned} \langle \Omega | \mathbf{x}_{m-k,Q}^+ \mathbf{x}_{k,Q}^- | \Omega \rangle &= (\Lambda_0^Q)^{-2} \sum_{j=0}^{m-k} \sum_{n=0}^{k-1} S_{k-1-n}^Q \Lambda_{n+1+j}^Q \\ &\quad \times \sum_{\ell'=0}^{m-k-j} (n+1+j-\ell') S_{m-k-j-\ell'}^Q \Lambda_{\ell'}^Q \\ &= (\Lambda_0^Q)^{-1} \left[ \sum_{j=0}^{m-k} \sum_{n=0}^{k-1} S_{k-1-n}^Q \Lambda_{n+1+j}^Q (n+1+j) \delta_{m,k+j} \right. \\ &\quad \left. - \sum_{j=0}^{m-k-1} \sum_{n=0}^{k-1} S_{k-1-n}^Q \Lambda_{n+1+j}^Q d_{m-k-j,Q} \right]. \end{aligned} \quad (80)$$

We then let  $n \rightarrow k-1-n$  and  $j \rightarrow m-k-j$ , resulting in

$$\begin{aligned} \langle \Omega | \mathbf{x}_{m-k,Q}^+ \mathbf{x}_{k,Q}^- | \Omega \rangle &= (\Lambda_0^Q)^{-1} \left[ \sum_{n=0}^{k-1} S_n^Q \Lambda_{m-n}^Q (m-n) - \sum_{j=1}^{m-k} \sum_{n=0}^{k-1} S_n^Q \Lambda_{m-j-n}^Q d_{j,Q} \right] \\ &= (\Lambda_0^Q)^{-1} \left[ \Lambda_0^Q d_{m,Q} - \sum_{n=k}^m S_n^Q \Lambda_{m-n}^Q (m-n) \right. \\ &\quad \left. - \sum_{j=1}^{m-k} d_{j,Q} \left( \delta_{m,j} - \sum_{n=k}^{m-j} S_n^Q \Lambda_{m-j-n}^Q \right) \right] = d_{m,Q}, \end{aligned} \quad (81)$$

where (74) is used for the first sum and (69) for the second sum. Since  $k > 1$ , we find  $\delta_{n,j} = 0$  for  $1 \leq j \leq n-k$ . Finally, (77) is used to show that (78) holds for  $1 \leq k \leq m$ , but not for  $k < 1$  when  $Q \neq 0$ , as the  $\mathbf{x}_{k,Q}^- | \Omega \rangle$  in (67) are defined only for  $k \geq 1$ , while the  $\langle \Omega | \mathbf{x}_{k,Q}^+$  in (68) are given for  $k \geq 0$ . Thus, this shows that only a subalgebra may exist, like those discussed in [13].

### 3.5. Generators of the quantum loop subalgebra

Formulae (67) for  $n = 1$  and (68) for  $n = 0$  suggest that

$$\mathbf{x}_{1,Q}^- = (\omega^Q / \Lambda_0^Q) \bar{\mathbf{C}}_0^{(Q)} \bar{\mathbf{B}}_1^{(N+Q)}, \quad \mathbf{x}_{0,Q}^+ = (\omega^Q / \Lambda_0^Q) \bar{\mathbf{C}}_0^{(N+Q)} \bar{\mathbf{B}}_1^{(Q)}, \quad (82)$$

and

$$\mathbf{h}_{1,Q} = [\mathbf{x}_{0,Q}^+, \mathbf{x}_{1,Q}^-], \quad \mathbf{x}_{n+2,Q}^- = \frac{1}{2}[\mathbf{h}_{1,Q}, \mathbf{x}_{n+1,Q}^-], \quad \mathbf{x}_{n+1,Q}^+ = -\frac{1}{2}[\mathbf{h}_{1,Q}, \mathbf{x}_{n,Q}^+], \quad (83)$$

for  $0 \leq n \leq \infty$ . Because of the complex form of these operators, to prove the Serre relations

$$[[[\mathbf{x}_{0,Q}^+, \mathbf{x}_{1,Q}^-], \mathbf{x}_{1,Q}^-], \mathbf{x}_{1,Q}^-] = 0, \quad [\mathbf{x}_{0,Q}^+, [\mathbf{x}_{0,Q}^+, [\mathbf{x}_{0,Q}^+, \mathbf{x}_{1,Q}^-]]] = 0 \quad (84)$$

is highly nontrivial. We can prove by induction the following,

$$(\mathbf{x}_{1,Q}^-)^n |\Omega\rangle = n!(\omega^Q/\Lambda_0^Q) \bar{\mathbf{C}}_0^{(Q)} \bar{\mathbf{B}}_1^{(nN+Q)} |\Omega\rangle, \quad 1 \leq n \leq m_Q, \quad (85)$$

$$(\mathbf{x}_{0,Q}^+)^m (\mathbf{x}_{1,Q}^-)^n |\Omega\rangle = m!n!(\omega^Q/\Lambda_0^Q) \bar{\mathbf{C}}_0^{(mN+Q)} \bar{\mathbf{B}}_1^{(nN+Q)} |\Omega\rangle, \quad 0 \leq m \leq n \leq m_Q. \quad (86)$$

The proofs are left to Appendix A.

These relations can be used to show that the first Serre relation in (84) holds for  $(\mathbf{x}_{1,Q}^-)^n |\Omega\rangle$ . That is

$$[[[\mathbf{x}_{0,Q}^+, \mathbf{x}_{1,Q}^-], \mathbf{x}_{1,Q}^-] \mathbf{x}_{1,Q}^-] (\mathbf{x}_{1,Q}^-)^n |\Omega\rangle = 0. \quad (87)$$

These details are in Appendix B. We managed to show that it also holds for  $(\mathbf{x}_{0,Q}^+)(\mathbf{x}_{1,Q}^-)^n |\Omega\rangle$ , but we have been unable to prove it for  $(\mathbf{x}_{0,Q}^+)^m (\mathbf{x}_{1,Q}^-)^n |\Omega\rangle$  for  $m > 1$ .

For general states  $|\{n_j\}\rangle$  satisfying the cyclic boundary condition  $n_1 + \dots + n_L \equiv 0 \pmod{N}$ , we again tested the Serre relation for small systems on a computer using Maple. The simplest nontrivial cases are  $N = 3$ ,  $L = 6$  and  $n_1 + \dots + n_6 = 3$ . Yet compared with the case  $Q = 0$ , the complexity increases enormously; each case, running in Maple 12 on ANU computers in Theoretical Physics takes five days. We have found that the Serre relation holds for the cases tested. Even though a formal proof is still lacking, we believe that the Serre relation (84) holds. As a consequence, we believe that the following loop subalgebra holds,

$$\begin{aligned} \mathbf{h}_{n,Q} &= [\mathbf{x}_{n-k,Q}^+, \mathbf{x}_{k,Q}^-], \quad \text{for } 1 \leq k \leq n, \\ \mathbf{x}_{n+k+1,Q}^- &= \frac{1}{2}[\mathbf{h}_{n,Q}, \mathbf{x}_{k+1,Q}^-], \quad \mathbf{x}_{n+k,Q}^+ = -\frac{1}{2}[\mathbf{h}_{n,Q}, \mathbf{x}_{k,Q}^+], \quad n > 1, k > 0. \end{aligned} \quad (88)$$

Since the indices here are nonnegative integers only, this is not the entire loop algebra, but a subalgebra as in [13].

### 3.6. Generators of the $\mathfrak{sl}_2$ algebra

In (3.2), the generators  $\mathbf{E}_{m,Q}^\pm$  on the ground state were given, but this is not sufficient. We can now define them in terms of generators of the loop algebra as in (II. 13), namely

$$\mathbf{E}_{m,Q}^+ = \sum_{n=0}^{r-1} \beta_{m^*,n}^Q z_{m,Q} \mathbf{x}_{n+1,Q}^-, \quad \mathbf{E}_{m,Q}^- = -\sum_{n=0}^{r-1} \beta_{m^*,n}^Q \mathbf{x}_{n,Q}^+. \quad (89)$$

The difference in the two equations is due to the fact that  $\mathbf{x}_{n,Q}^-$  is defined for  $n \geq 1$ , while  $\mathbf{x}_{n,Q}^+$  is defined for  $n \geq 0$ . The commutators are

$$\mathbf{H}_{m,Q} = [\mathbf{E}_{m,Q}^+, \mathbf{E}_{m,Q}^-] = \sum_{n=0}^{r-1} \beta_{m^*,n} z_{m,Q} \mathbf{h}_{n+1,Q}. \quad (90)$$

Because of equation (86), we may rewrite (57) as

$$\mathbf{E}_{m,Q}^+|\Omega\rangle = \beta_{m,0}^Q \sum_{\ell=1}^{m_Q} z_{m,Q}^\ell (\mathbf{x}_{0,Q}^+)^{(\ell-1)} (\mathbf{x}_{1,Q}^-)^{(\ell)} |\Omega\rangle, \quad (\mathbf{x}_{n,Q}^\pm)^{(\ell)} \equiv \frac{(\mathbf{x}_{n,Q}^\pm)^\ell}{\ell!}. \quad (91)$$

Assuming that the Serre relation (84) holds, we may again prove by induction

$$\begin{aligned} [(\mathbf{x}_{0,Q}^+)^{(j)}, (\mathbf{x}_{k,Q}^-)] &= (\mathbf{x}_{0,Q}^+)^{(j-1)} \mathbf{h}_{k,Q} - (\mathbf{x}_{k,Q}^+)(\mathbf{x}_{0,Q}^+)^{(j-2)}, \\ [(\mathbf{x}_{k,Q}^+), (\mathbf{x}_{1,Q}^-)^{(j)}] &= (\mathbf{x}_{1,Q}^-)^{(j-1)} \mathbf{h}_{k+1,Q} + (\mathbf{x}_{k+2,Q}^-)(\mathbf{x}_{1,Q}^-)^{(j-2)}, \\ [\mathbf{h}_{k,Q}, (\mathbf{x}_{0,Q}^+)^{(j)}] &= -2(\mathbf{x}_{k,Q}^+)(\mathbf{x}_{0,Q}^+)^{(j-1)}, \quad [\mathbf{h}_{k,Q}, (\mathbf{x}_{1,Q}^-)^{(j)}] = 2(\mathbf{x}_{k+1,Q}^-)(\mathbf{x}_{1,Q}^-)^{(j-1)}, \end{aligned} \quad (92)$$

so that Appendix B in [7] can be repeated here to show that

$$\begin{aligned} \mathbf{E}_{j,Q}^+ \mathbf{E}_{m,Q}^+ |\Omega\rangle &= \beta_{m,0}^Q \left\{ (1 - z_{m,Q}/z_{j,Q})^2 \sum_{\ell=1}^{r-1} z_{m,Q}^\ell (\mathbf{x}_{0,Q}^+)^{(\ell-1)} (\mathbf{x}_{1,Q}^-)^{(\ell)} \mathbf{E}_j^+ |\Omega\rangle \right. \\ &\quad \left. + (1 - z_{m,Q}/z_{j,Q}) \sum_{\ell=2}^r z_{m,Q}^\ell (\mathbf{x}_{0,Q}^+)^{(\ell-2)} (\mathbf{x}_{1,Q}^-)^{(\ell)} |\Omega\rangle \right\}. \end{aligned} \quad (93)$$

Consequently, we again have  $(\mathbf{E}_{m,Q}^+)^2 |\Omega\rangle = 0$ .

If we let

$$\mathbf{x}_{0,Q}^- = (\Lambda_0^Q)^{-1} \bar{\mathbf{C}}_{L-1}^{(Q)} \bar{\mathbf{B}}_L^{(N+Q)}, \quad \mathbf{x}_{-1,Q}^+ = (\Lambda_0^Q)^{-1} \bar{\mathbf{C}}_{L-1}^{(N+Q)} \bar{\mathbf{B}}_L^{(Q)}, \quad (94)$$

so that for  $Q = 0$  we have  $\mathbf{x}_{0,Q}^- \rightarrow \mathbf{x}_0^-$  and  $\mathbf{x}_{-1,Q}^+ \rightarrow \mathbf{x}_{-1}^+$ , one can see from (45) that  $\mathbf{x}_{0,Q}^-$  and  $\mathbf{x}_{-1,Q}^+$  are not eigenvectors of  $\tau_2(t_q)|_Q$ , but eigenvectors of  $\tau_2(t_q)|_{-Q}$ . However,

$$\mathbf{x}_{0,N-Q}^- = (\Lambda_0^{N-Q})^{-1} \bar{\mathbf{C}}_{L-1}^{(N-Q)} \bar{\mathbf{B}}_L^{(2N-Q)}, \quad \mathbf{x}_{-1,N-Q}^+ = (\Lambda_0^{N-Q})^{-1} \bar{\mathbf{C}}_{L-1}^{(2N-Q)} \bar{\mathbf{B}}_L^{(N-Q)}, \quad (95)$$

are eigenvectors of  $\tau_2(t_q)|_Q$ , but corresponding to the Drinfeld polynomial  $P_{N-Q}(z)$ . It is also possible to express  $\mathbf{E}_{j,Q}^\pm$  in terms of these operators.

#### 4. Transfer Matrix Eigenvectors

From (48) we see that  $2^{r-1}$  eigenvectors obeying (46) can be generated by operating the  $r-1$  operators  $\mathbf{E}_{j,Q}^+$  on the ground state  $|\Omega\rangle$ , while  $2^{r-1}$  eigenvectors satisfying (47) are found by operating the  $r-1$  operators  $\bar{\mathbf{E}}_{j,N-Q}^-$  on  $|\bar{\Omega}\rangle$ . The  $\bar{\mathbf{E}}_{j,N-Q}^-$  differ from the  $\mathbf{E}_{j,N-Q}^-$  in that the positions of the  $\bar{\mathbf{B}}_1$  and  $\bar{\mathbf{C}}_0$  are interchanged, as can be seen from (48) and (49).

We now show how the transfer matrices can be expressed in terms of these generators and how their eigenvectors can be obtained.

##### 4.1. Ground state sector eigenvalues

From (6.2) and (6.14) of Baxter [2], we find

$$\mathcal{T}_Q(x_q, y_q) \mathbf{x} = x_q^{P_a} y_q^{P_b} \mathcal{G}(\lambda_q) \mathbf{y}, \quad (96)$$

where  $P_a = Q$  and  $P_b = 0$  for the  $2^{r-1}$  eigenvectors satisfying (38), while  $P_a = 0$  and  $P_b = N - Q$  for the  $2^{r-1}$  eigenvectors obeying (39). Thus we have

$$\mathcal{G}_a(\lambda_q)\mathcal{G}_a(\lambda_q^{-1}) = Nt_p^{rN}P_Q(t^N) = Nt_p^{rN}\Lambda_{m_Q}^Q \prod_{j=1}^{r-1}[(t_q/t_p)^N - z_{j,Q}], \quad (97)$$

for the former case, and

$$\mathcal{G}_b(\lambda_q)\mathcal{G}_b(\lambda_q^{-1}) = \omega^Q Nt_p^{rN}P_{N-Q}(t^N) = \omega^Q Nt_p^{rN}\Lambda_0^Q \prod_{j=1}^{r-1}[(t_q/t_p)^N - z_{j,Q}^{-1}] \quad (98)$$

for the latter. Here subscripts  $a$  and  $b$  have been inserted to distinguish the two cases. Because  $\Lambda_{m_Q-j}^Q = \Lambda_j^{N-Q}$ , the roots of  $P_{N-Q}(z)$  are the inverses of the roots of  $P_Q(z)$ . Consequently, as in (II.8), we may write

$$\mathcal{G}_a(\lambda_q) = D_Q \prod_{j=1}^{r-1} (A_{j,Q} \pm B_{j,Q}), \quad \mathcal{G}_b(\lambda_q) = \hat{D}_Q \prod_{j=1}^{r-1} (A_{j,N-Q} \pm B_{j,N-Q}), \quad (99)$$

where

$$A_{j,Q} = \cosh \theta_{j,Q}(1 - \lambda_q^{-1}), \quad B_{j,Q} = \sinh \theta_{j,Q}(1 + \lambda_q^{-1}), \quad (100)$$

$$D_Q = (Nt_p^N \Lambda_0^{N-Q})^{\frac{1}{2}} (k'/k^2)^{\frac{1}{2}r}, \quad \hat{D}_Q = (\omega^Q Nt_p^{rN} \Lambda_0^Q)^{\frac{1}{2}} (k'/k^2)^{\frac{1}{2}r}, \quad (101)$$

with  $\theta_{j,Q}$  given by (II.6) replacing  $z_j \rightarrow z_{j,Q}$ , i.e.

$$2 \cosh 2\theta_{j,Q} = k' + k'^{-1} - k^2 t_p^N z_{j,Q}/k', \quad \theta_{j,N-Q} = \theta_{j^*,Q}, \quad z_{j,Q} z_{j^*,Q} = 1. \quad (102)$$

We have also changed  $A_{j,Q}$  compared with [7] by dropping the constant  $\rho$ , but absorbing it into the constant  $D_Q$ .

#### 4.2. Eigenvectors corresponding to $x_q^Q \mathcal{G}_a(\lambda_q)$

We consider first the eigenvectors of the transfer matrix related to (38) and (97). Similar to (II.100), we may write

$$\begin{aligned} \mathcal{T}_Q(x_q, y_q) &= x_q^Q D_Q \prod_{j=1}^{r-1} [X_{j,Q} - \mathbf{H}_{j,Q} Y_{j,Q} + (\mathbf{E}_{j,Q}^+ + \mathbf{E}_{j,Q}^-) Z_{j,Q}] \\ &= x_q^Q D_Q \prod_{j=1}^{r-1} \mathcal{S}_{j,Q} [A_{j,Q} - \mathbf{H}_{j,Q} B_{j,Q}] \mathcal{R}_{j,Q}^{-1}, \end{aligned} \quad (103)$$

so that the  $2^{r-1}$  sought eigenvectors of the transfer matrix are given by

$$|\mathcal{X}_i^a\rangle = \prod_{j=1}^{r-1} \mathcal{R}_{j,Q} \prod_{m \in J_n} \mathbf{E}_{m,Q}^+ |\Omega\rangle, \quad |\mathcal{Y}_i^a\rangle = \prod_{j=1}^{r-1} \mathcal{S}_{j,Q} \prod_{m \in J_n} \mathbf{E}_{m,Q}^+ |\Omega\rangle. \quad (104)$$

Here  $J_n = \{j_1, \dots, j_n\}$ , for  $0 \leq n \leq r-1$ , is any subset of  $\{1, 2, \dots, r-1\}$ , such that

$$\mathcal{T}_Q(x_q, y_q) |\mathcal{X}_i^a\rangle = x_q^Q D_Q \prod_{j=1}^{r-1} [A_{j,Q} \pm B_{j,Q}] |\mathcal{Y}_i^a\rangle. \quad (105)$$

To evaluate  $\mathcal{R}_{j,Q}$ , and  $\mathcal{S}_{j,Q}$ , we use (II.39) to find

$$\langle \Omega | \mathcal{T}_Q(x_q, y_q) | \Omega \rangle = N^{1-\frac{1}{2}L} y_p^{rN} (x_q/y_p)^Q P_Q(x_q^N/y_p^N). \quad (106)$$

We then use (II.37) and (II.63) to obtain

$$\langle \{n_j\} | \mathcal{T}_Q(x_q, y_q) | \Omega \rangle = N^{1-\frac{1}{2}L} \omega^{-\sum_j j n_j} y_p^{rN} (1 - x_q^N/y_p^N) G_Q(\{n_j\}, x_q^N/y_p^N). \quad (107)$$

From (61) and (III.43), we find

$$\langle \Omega | \mathbf{E}_{m,Q}^- \mathcal{T}_Q(x_q, y_q) | \Omega \rangle = -(\beta_{m,0}^Q/\Lambda_0^Q) N^{1-\frac{1}{2}L} y_p^{rN} (1 - x_q^N/y_p^N) (x_q/y_p)^Q \mathbf{h}_m^Q(x_q^N/y_p^N). \quad (108)$$

Consequently, (III.54), (1) and (63) can be used to get the ratio

$$\frac{\langle \Omega | \mathcal{T}_Q(x_q, y_q) | \Omega \rangle}{\langle \Omega | \mathbf{E}_{m,Q}^- \mathcal{T}_Q(x_q, y_q) | \Omega \rangle} = \frac{x_q^N - y_p^N z_{m,Q}}{x_q^N - y_p^N} = \frac{X_{m,Q} + Y_{m,Q}}{Z_{m,Q}}. \quad (109)$$

Again, as in [7], the ratio depends on  $z_{m,Q}$  only, so that  $\mathcal{R}_{j,Q}$  and  $\mathcal{S}_{j,Q}$  are independent of the other roots of  $P_Q(z)$ . Since  $|\bar{\Omega}\rangle$  has an eigenvalue different from the one of  $|\Omega\rangle$ , we cannot evaluate the other ratio as in [7]. However, as

$$\begin{aligned} \hat{\mathcal{T}}_Q(y_q, x_q) &= y_q^Q D_Q \prod_{j=1}^{r-1} [\bar{X}_{j,Q} - \mathbf{H}_{j,Q} \bar{Y}_{j,Q} + (\mathbf{E}_{j,Q}^+ + \mathbf{E}_{j,Q}^-) \bar{Z}_{j,Q}] \\ &= y_q^Q D_Q \prod_{j=1}^{r-1} \mathcal{R}_{j,Q} (\bar{A}_{j,Q} - \mathbf{H}_{j,Q} \bar{B}_{j,Q}) \mathcal{S}_{j,Q}^{-1}, \end{aligned} \quad (110)$$

with  $\bar{A}_{j,Q}$  and  $\bar{B}_{j,Q}$  obtained from  $A_{j,Q}$  and  $B_{j,Q}$  in (100) replacing  $\lambda_q^{-1}$  by  $\lambda_q$ , we may use (II.95), (1) and (II.64) to find

$$\langle \Omega | \hat{\mathcal{T}}_Q(y_q, x_q) | \Omega \rangle = N^{1-\frac{1}{2}L} x_p^{rN} (y_q/x_p)^Q P_Q(y_q^N/x_p^N) \quad (111)$$

and

$$\langle \Omega | \hat{\mathcal{T}}_Q(y_q, x_q) | \{n_j\} \rangle = N^{1-\frac{1}{2}L} \omega^{\sum_j j n_j} x_p^{rN} (1 - y_q^N/x_p^N) \bar{G}_Q(\{n_j\}, y_q^N/x_p^N). \quad (112)$$

Next we use (62), (III.55) and (III.56) to derive

$$\langle \Omega | \hat{\mathcal{T}}_Q(y_q, x_q) \mathbf{E}_{m,Q}^+ | \Omega \rangle = -(\beta_{m,0}^Q/\Lambda_0^Q) N^{1-\frac{1}{2}L} x_p^{rN} (1 - y_q^N/x_p^N) (y_q/x_p)^Q \bar{\mathbf{h}}_m^Q(y_q^N/x_p^N), \quad (113)$$

so that we can evaluate the second ratio as

$$\frac{\langle \Omega | \hat{\mathcal{T}}_Q(y_q, x_q) | \Omega \rangle}{\langle \Omega | \hat{\mathcal{T}}_Q(y_q, x_q) \mathbf{E}_{m,Q}^+ | \Omega \rangle} = -\frac{x_p^N - y_q^N z_{m,Q}^{-1}}{x_p^N - y_q^N} = \frac{\bar{X}_{j,Q} + \bar{Y}_{j,Q}}{\bar{Z}_{j,Q}}. \quad (114)$$

The  $j$ th term in the product of (110) yields

$$[\bar{X}_{j,Q} - \mathbf{H}_{j,Q} \bar{Y}_{j,Q} + (\mathbf{E}_{j,Q}^+ + \mathbf{E}_{j,Q}^-) \bar{Z}_{j,Q}] = \mathcal{R}_{j,Q} [\bar{A}_{j,Q} - \mathbf{H}_{j,Q} \bar{B}_{j,Q}] \mathcal{S}_{j,Q}^{-1}. \quad (115)$$

Choosing the determinants of  $\mathcal{R}_{j,Q}$  and  $\mathcal{S}_{j,Q}$  to be one, we find

$$(\bar{X}_{j,Q}^2 - \bar{Y}_{j,Q}^2 - \bar{Z}_{j,Q}^2) = (\bar{A}_{j,Q}^2 - \bar{B}_{j,Q}^2). \quad (116)$$

By inverting both sides of (115), and using (116), we express (115) in the diagonal representation of  $\mathbf{H}_{j,Q}$  as

$$\begin{bmatrix} \bar{X}_{j,Q} + \bar{Y}_{j,Q} & -\bar{Z}_{j,Q} \\ -\bar{Z}_{j,Q} & \bar{X}_{j,Q} - \bar{Y}_{j,Q} \end{bmatrix} = \mathcal{S}_{j,Q} \begin{bmatrix} e^{\theta_{j,Q}} - \lambda_q e^{-\theta_{j,Q}} & 0 \\ 0 & e^{-\theta_{j,Q}} - \lambda_q e^{\theta_{j,Q}} \end{bmatrix} \mathcal{R}_{j,Q}^{-1}. \quad (117)$$

Similarly the  $j$ th term in the product in (103) can be written as

$$\begin{bmatrix} X_{j,Q} - Y_{j,Q} & Z_{j,Q} \\ Z_{j,Q} & X_{j,Q} + Y_{j,Q} \end{bmatrix} = \mathcal{S}_{j,Q} \begin{bmatrix} e^{-\theta_{j,Q}} - \lambda_q^{-1} e^{\theta_{j,Q}} & 0 \\ 0 & e^{\theta_{j,Q}} - \lambda_q^{-1} e^{-\theta_{j,Q}} \end{bmatrix} \mathcal{R}_{j,Q}^{-1}. \quad (118)$$

It is easy to see that (109) gives the lower right triangle in the left hand side of (118) except for a constant  $\epsilon_{j,Q}$ , that is

$$\begin{aligned} X_{j,Q} + Y_{j,Q} &= \epsilon_{j,Q} k(y_p^N z_{j,Q} - x_q^N) = \epsilon_{j,Q} [(1 - k' \lambda_p) z_{j,Q} - (1 - k' \lambda_q^{-1})], \\ Z_{j,Q} &= \epsilon_{j,Q} k(y_p^N - x_q^N) = \epsilon_{j,Q} k'(\lambda_q^{-1} - \lambda_p), \end{aligned} \quad (119)$$

while (114) determines the upper left triangle in (117) except for a constant  $\bar{\epsilon}_{j,Q}$ , i.e.

$$\begin{aligned} \bar{X}_{j,Q} + \bar{Y}_{j,Q} &= \bar{\epsilon}_{j,Q} k(x_p^N - y_q^N z_{j,Q}^{-1}) = \bar{\epsilon}_{j,Q} [(1 - k' \lambda_p^{-1}) - (1 - k' \lambda_q) z_{j,Q}^{-1}], \\ \bar{Z}_{j,Q} &= \bar{\epsilon}_{j,Q} k(x_p^N - y_q^N) = \bar{\epsilon}_{j,Q} k'(\lambda_q - \lambda_p^{-1}). \end{aligned} \quad (120)$$

The matrices in (118) are linear in  $\lambda_q^{-1}$ , while those in (117) are linear in  $\lambda_q$ . Thus by equating the constant and linear terms, and letting  $\epsilon_{j,Q} \lambda_p = \bar{\epsilon}_{j,Q}$ , we find equations identical to those in (II.90), namely

$$\mathcal{S}_{j,Q} \begin{bmatrix} e^{-\theta_{j,Q}} & 0 \\ 0 & e^{\theta_{j,Q}} \end{bmatrix} \mathcal{R}_{j,Q}^{-1} = \mathbf{M}, \quad \mathcal{S}_{j,Q} \begin{bmatrix} e^{\theta_{j,Q}} & 0 \\ 0 & e^{-\theta_{j,Q}} \end{bmatrix} \mathcal{R}_{j,Q}^{-1} = -\mathbf{N}, \quad (121)$$

with the matrix elements of  $\mathbf{M}$  and  $\mathbf{N}$  almost identical to (II.91),

$$\begin{aligned} m_{11} &= -\epsilon_{j,Q} k' \lambda_p / z_{j,Q}, \quad m_{12} = m_{21} = -\epsilon_{j,Q} k' \lambda_p, \quad m_{22} = \epsilon_{j,Q} (z_{j,Q} - 1 - k' z_{j,Q} \lambda_p), \\ n_{11} &= \epsilon_{j,Q} (z_{j,Q}^{-1} \lambda_p - \lambda_p + k'), \quad n_{12} = n_{21} = n_{22} = \epsilon_{j,Q} k'. \end{aligned} \quad (122)$$

Evaluating the determinants of both sides of (121), we again find  $\epsilon_{j,Q}^2 k' ((z_{j,Q}^{-1} - 1) \lambda_p = 1$ . Consequently, the matrices  $\mathcal{S}_{j,Q}$  and  $\mathcal{R}_{j,Q}$  can be evaluated in exactly the same way as in [7], with the result

$$\mathcal{S}_{j,Q} = \frac{1}{2}(s_{11} + s_{22})\mathbf{1} + \frac{1}{2}(s_{11} - s_{22})\mathbf{H}_{j,Q} + s_{12}\mathbf{E}_{j,Q}^+ + s_{21}\mathbf{E}_{j,Q}^-, \quad (123)$$

$$\mathcal{R}_{j,Q} = \frac{1}{2}(r_{11} + r_{22})\mathbf{1} + \frac{1}{2}(r_{11} - r_{22})\mathbf{H}_{j,Q} + r_{12}\mathbf{E}_{j,Q}^+ + r_{21}\mathbf{E}_{j,Q}^-, \quad (124)$$

where

$$\begin{aligned} s_{22} = r_{11} &= \left( \frac{m_{22} e^{\theta_{j,Q}} + n_{22} e^{-\theta_{j,Q}}}{2 \sinh 2\theta_{j,Q}} \right)^{\frac{1}{2}}, \quad s_{12} = -r_{21} = \left( \frac{m_{12} e^{\theta_{j,Q}} + n_{12} e^{-\theta_{j,Q}}}{m_{22} e^{\theta_{j,Q}} + n_{22} e^{-\theta_{j,Q}}} \right) s_{22}, \\ s_{21} = -r_{12} &= \frac{e^{-2\theta_{j,Q}} - k'}{2 \sinh 2\theta_{j,Q} s_{12}}, \quad s_{11} = r_{22} = \frac{e^{2\theta_{j,Q}} - k'}{2 \sinh 2\theta_{j,Q} s_{22}}. \end{aligned} \quad (125)$$

#### 4.3. Eigenvectors corresponding to $y_q^{N-Q} \mathcal{G}_b(\lambda_q)$

We now consider eigenvectors of the transfer matrix related to (39) and (98). From (49), we find that the generators of  $\mathfrak{sl}_2$  algebra can be written, similar to (56) and (57), as

$$\langle \bar{\Omega} | \bar{\mathbf{E}}_{m,Q}^+ = \omega^{Q(Q+1)} (\beta_{m,0}^Q / \Lambda_0^Q) \sum_{\ell=1}^{m_Q} z_{m,Q}^\ell \langle \bar{\Omega} | \bar{\mathbf{B}}_1^{(\ell N+Q)} \bar{\mathbf{C}}_0^{(\ell N-N+Q)}, \quad (126)$$

$$\bar{\mathbf{E}}_{m,Q}^- | \bar{\Omega} \rangle = -\omega^{Q(Q+1)} (\beta_{m,0}^Q / \Lambda_0^Q) \sum_{\ell=1}^{m_Q} z_{m,Q}^{\ell-1} \bar{\mathbf{B}}_1^{(\ell N-N+Q)} \bar{\mathbf{C}}_0^{(\ell N+Q)} | \bar{\Omega} \rangle, \quad (127)$$



which are generalizations of the second equations in (II.53) and (II.54). Similar to the derivation of (61) and (62), we generalize (II.67) and (II.69) to

$$\langle \bar{\Omega} | \bar{\mathbf{E}}_{m,Q}^+ = -(\beta_{m,0}^Q / \Lambda_0^Q) z_{m,Q} \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \langle \{N-1-n_j\} | \bar{G}_Q(\{n_j\}, z_{m,Q}), \quad (128)$$

$$\bar{\mathbf{E}}_{k,Q}^- | \bar{\Omega} \rangle = (\beta_{k,0}^Q / \Lambda_0^Q) \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} G_Q(\{n_j\}, z_{k,Q}) | \{N-1-n_j\} \rangle. \quad (129)$$

Again, we use the theorem in [8] to find that

$$\langle \bar{\Omega} | \bar{\mathbf{E}}_{m,Q}^+ \bar{\mathbf{E}}_{m,Q}^- | \bar{\Omega} \rangle = \langle \bar{\Omega} | \bar{\mathbf{H}}_{m,Q} | \bar{\Omega} \rangle = 1. \quad (130)$$

Comparing these results with those for  $\mathbf{E}_{m,Q}^\pm$  in (61) and (62), we can see that what was done in subsection 3.5 can be repeated here to obtain the generators of a different quantum loop subalgebra, with generators  $\bar{\mathbf{E}}_{m,Q}^\pm$  as in (89) and  $\bar{\mathbf{H}}_{m,Q}$  as in (90).

Using (96) and (102), we may write

$$\begin{aligned} \mathcal{T}_Q(x_q, y_q) &= y_q^{N-Q} \hat{D}_Q \prod_{j=1}^{r-1} [X_{j^*,Q} - \bar{\mathbf{H}}_{j,N-Q} Y_{j^*,Q} + (\bar{\mathbf{E}}_{j,N-Q}^+ + \bar{\mathbf{E}}_{j,N-Q}^-) Z_{j^*,Q}] \\ &= y_q^{N-Q} \hat{D}_Q \prod_{j=1}^{r-1} \mathcal{S}_{j^*,Q} (A_{j^*,Q} - \mathbf{H}_{j,N-Q} B_{j^*,Q}) \mathcal{R}_{j^*,Q}^{-1}, \end{aligned} \quad (131)$$

so that the  $2^{r-1}$  eigenvectors of the transfer matrix are given by

$$|\mathcal{X}_i^b\rangle = \prod_{j=1}^{r-1} \mathcal{R}_{j^*,Q} \prod_{m \in J_n} \bar{\mathbf{E}}_{m,N-Q}^- | \bar{\Omega} \rangle, \quad |\mathcal{Y}_i^b\rangle = \prod_{j=1}^{r-1} \mathcal{S}_{j^*,Q} \prod_{m \in J_n} \bar{\mathbf{E}}_{m,N-Q}^- | \bar{\Omega} \rangle. \quad (132)$$

Here  $J_n = \{j_1, \dots, j_n\}$ , for  $0 \leq n \leq r-1$ , is any subset of  $\{1, 2, \dots, r-1\}$ , so that

$$\mathcal{T}_Q(x_q, y_q) |\mathcal{X}_i^b\rangle = y_q^{N-Q} \hat{D}_Q \prod_{j=1}^{r-1} (A_{j^*,Q} \pm B_{j^*,Q}) |\mathcal{Y}_i^b\rangle. \quad (133)$$

We may follow the procedure in subsection 4.2 to get

$$\mathcal{S}_{j^*,Q} = \frac{1}{2}(s'_{11} + s_{22}) \mathbf{1} + \frac{1}{2}(s'_{11} - s'_{22}) \bar{\mathbf{H}}_{j,N-Q} + s'_{12} \bar{\mathbf{E}}_{j,N-Q}^+ + s'_{21} \bar{\mathbf{E}}_{j,N-Q}^-, \quad (134)$$

$$\mathcal{R}_{j^*,Q} = \frac{1}{2}(r'_{11} + r'_{22}) \mathbf{1} + \frac{1}{2}(r'_{11} - r'_{22}) \bar{\mathbf{H}}_{j,N-Q} + r'_{12} \bar{\mathbf{E}}_{j,N-Q}^+ + r'_{21} \bar{\mathbf{E}}_{j,N-Q}^-, \quad (135)$$

where the  $s'_{ik}$  and  $r'_{ik}$  are again given by (125), but with the replacements  $z_{j,Q} \rightarrow z_{j,Q}^{-1}$ , and  $\theta_{j,Q} \rightarrow \theta_{j^*,Q}$ .

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**Appendix A. Identities (85) and (86)**

To prove (85) by induction, it is easily seen from (67) that it holds for  $n = 1$ . We now assume this is also true for  $n = m$ . Similar to (60), we use (50) and (II.52) to obtain

$$(\mathbf{x}_{1,Q}^-)^m |\Omega\rangle = \frac{m! \omega^Q}{\Lambda_0^Q} \bar{\mathbf{C}}_0^{(Q)} \bar{\mathbf{B}}_1^{(mN+Q)} |\Omega\rangle = \frac{m!}{\Lambda_0^Q} \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = mN}} \omega^{-\sum_j j n_j} K_Q(\{n_j\}) |\{n_j\}\rangle. \quad (\text{A.1})$$

Again, we use (50) and (II.52) to rewrite (82) as

$$\begin{aligned} \Lambda_0^Q \mathbf{x}_{1,Q}^- |\{n_j\}\rangle &= \sum_{\substack{\{0 \leq \mu_j, n'_j \leq N-1\} \\ \sum n'_j = mN+N \\ \sum \mu_j = Q}} \omega^{\sum_j j(n_j - n'_j)} \prod_{j=1}^L \begin{bmatrix} n'_j + \mu_j \\ \mu_j \end{bmatrix} \begin{bmatrix} n'_j + \mu_j \\ n_j \end{bmatrix} \\ &\quad \times \omega^{n_j(N'_j + a_j - N_j) + \mu_j N'_j} |\{n'_j\}\rangle, \end{aligned} \quad (\text{A.2})$$

and

$$\begin{aligned} \Lambda_0^Q \mathbf{x}_{0,Q}^+ |\{n_j\}\rangle &= \sum_{\substack{\{0 \leq \mu_j, n'_j \leq N-1\} \\ \sum n'_j = mN-N \\ \sum \mu_j = N+Q}} \omega^{\sum_j j(n_j - n'_j)} \prod_{j=1}^L \begin{bmatrix} n'_j + \mu_j \\ \mu_j \end{bmatrix} \begin{bmatrix} n'_j + \mu_j \\ n_j \end{bmatrix} \\ &\quad \times \omega^{n_j(N'_j + a_j - N_j) + \mu_j N'_j} |\{n'_j\}\rangle. \end{aligned} \quad (\text{A.3})$$

Here, as in our previous papers, we have defined

$$N_j \equiv \sum_{\ell < j} n_\ell, \quad N'_j \equiv \sum_{\ell < j} n'_\ell, \quad a_j \equiv \sum_{\ell < j} \mu_\ell. \quad (\text{A.4})$$

After multiplying (A.1) by  $\mathbf{x}_{1,Q}^-$  and using (A.2) together with (III.4) and (III.18), we find

$$\begin{aligned} (\mathbf{x}_{1,Q}^-)^{m+1} |\Omega\rangle &= \frac{m!}{(\Lambda_0^Q)^2} \sum_{\substack{\{0 \leq \lambda_j, \mu_j, n'_j \leq N-1\} \\ \sum n'_j = mN+N \\ \sum \mu_j = \sum \lambda_j = Q}} \omega^{-\sum_j j n'_j} I_{mN}(\{n'_j + \mu_j\}; \{\lambda_j\}) \\ &\quad \times \prod_{j=1}^L \begin{bmatrix} n'_j + \mu_j \\ \mu_j \end{bmatrix} \omega^{\mu_j N'_j} |\{n'_j\}\rangle \\ &= \frac{(m+1)!}{\Lambda_0^Q} \sum_{\substack{\{0 \leq n'_j \leq N-1\} \\ \sum n'_j = mN+N}} \omega^{-\sum_j j n'_j} K_Q(\{n'_j\}) |\{n'_j\}\rangle \\ &= \frac{(m+1)! \omega^Q}{\Lambda_0^Q} \bar{\mathbf{C}}_0^{(Q)} \bar{\mathbf{B}}_1^{(mN+N+Q)} |\Omega\rangle, \end{aligned} \quad (\text{A.5})$$

where (III.19) of Lemma 1 in [8] is used and also (51) and (III.4) to carry out the other two sums. This then proves (85). To prove (86), we first prove it for  $m = 1$ . After

multiplying (A.1) (in which  $m$  is replaced by  $n$ ) by  $\mathbf{x}_{0,Q}^+$ , and then using (A.3), we get

$$\begin{aligned} \mathbf{x}_{0,Q}^+(\mathbf{x}_{1,Q}^-)^n|\Omega\rangle &= \frac{n!}{(\Lambda_0^Q)^2} \sum_{\substack{\{0 \leq \lambda_j, \mu_j, n'_j \leq N-1\} \\ \sum n'_j = nN-N \\ \sum \lambda_j = Q, \sum \mu_j = N+Q}} \omega^{-\sum_j j n'_j} I_{nN}(\{n'_j + \mu_j\}; \{\lambda_j\}) \\ &\quad \times \prod_{j=1}^L \begin{bmatrix} n'_j + \mu_j \\ \mu_j \end{bmatrix} \omega^{\mu_j N'_j} |\{n'_j\}\rangle. \end{aligned} \quad (\text{A.6})$$

Again we use (III.19) of Lemma 1 in [8], then use (51) and (III.4) to obtain

$$\begin{aligned} \mathbf{x}_{0,Q}^+(\mathbf{x}_{1,Q}^-)^n|\Omega\rangle &= \frac{(n)!}{\Lambda_0^Q} \sum_{\substack{\{0 \leq n'_j \leq N-1\} \\ \sum n'_j = nN-N}} \omega^{-\sum_j j n'_j} K_{N+Q}(\{n'_j\}) |\{n'_j\}\rangle \\ &= \frac{(n)! \omega^Q}{\Lambda_0^Q} \bar{\mathbf{C}}_0^{(N+Q)} \bar{\mathbf{B}}_1^{(nN+Q)} |\Omega\rangle. \end{aligned} \quad (\text{A.7})$$

This shows that (86) holds for  $m = 1$ . Next assume that (86) holds for  $m = \ell$ , so that

$$(\mathbf{x}_{0,Q}^+)^{\ell} (\mathbf{x}_{1,Q}^-)^n |\Omega\rangle = \frac{\ell! n!}{\Lambda_0^Q} \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ \sum n_j = nN - \ell N}} \omega^{-\sum_j j n_j} K_{\ell N + Q}(\{n_j\}) |\{n_j\}\rangle. \quad (\text{A.8})$$

To prove that it also holds for  $m = \ell + 1$ , we multiply (A.8) by  $\mathbf{x}_{0,Q}^+$  and then use (A.3), together with (III.4) and (III.18), to find for  $\sum \lambda_j = \ell N + Q$ , and  $\sum \mu_j = N + Q$ ,

$$\begin{aligned} &[(\Lambda_0^Q)^2 / \ell! n!] (\mathbf{x}_{0,Q}^+)^{\ell+1} (\mathbf{x}_{1,Q}^-)^n |\Omega\rangle \\ &= \sum_{\substack{\{0 \leq \lambda_j, \mu_j, n'_j \leq N-1\} \\ \sum n'_j = nN - \ell N - N}} \omega^{-\sum_j j n'_j} I_{nN - \ell N}(\{n'_j + \mu_j\}; \{\lambda_j\}) \\ &\quad \times \prod_{j=1}^L \begin{bmatrix} n'_j + \mu_j \\ \mu_j \end{bmatrix} \omega^{\mu_j N'_j} |\{n'_j\}\rangle \\ &= \sum_{\substack{\{0 \leq \lambda_j, \mu_j, n'_j \leq N-1\} \\ \sum n'_j = nN - \ell N - N}} \omega^{-\sum_j j n'_j} \bar{I}_{\ell N}(\{\lambda_j\}; \{n'_j + \mu_j\}) \\ &\quad \times \prod_{j=1}^L \begin{bmatrix} n'_j + \mu_j \\ \mu_j \end{bmatrix} \omega^{\mu_j N'_j} |\{n'_j\}\rangle, \end{aligned} \quad (\text{A.9})$$

where (III.21) is used. Using (III.23) and the identity,

$$\begin{bmatrix} n'_j + \mu_j \\ \mu_j \end{bmatrix} \begin{bmatrix} n'_j + n_j + \mu_j \\ n_j \end{bmatrix} = \begin{bmatrix} n_j + \mu_j \\ n_j \end{bmatrix} \begin{bmatrix} n'_j + n_j + \mu_j \\ n_j + \mu_j \end{bmatrix}, \quad \sum_{j=1}^L n_j = \ell N, \quad (\text{A.10})$$

we may rewrite (A.9), by making the change of variables  $\mu'_j = n_j + \mu_j$  and  $\lambda_j = \lambda'_j + n_j$  with  $\mu'_1 + \dots + \mu'_L = \ell N + N + Q$  and  $\lambda'_1 + \dots + \lambda'_L = Q$ , as

$$\begin{aligned}
& [(\Lambda_0^Q)^2 / \ell! n!] (\mathbf{x}_{0,Q}^+)^{\ell+1} (\mathbf{x}_{1,Q}^-)^n |\Omega\rangle \\
&= \sum_{\substack{\{0 \leq \lambda'_j, \mu'_j, n'_j \leq N-1\} \\ \sum n'_j = nN - \ell N - N}} \omega^{-\sum_j j n'_j} I_{\ell N}(\{\mu'_j\}; \{\lambda'_j\}) \\
&\quad \times \prod_{j=1}^L \begin{bmatrix} n'_j + \mu'_j \\ \mu'_j \end{bmatrix} \omega^{\mu'_j N'_j} |\{n'_j\}\rangle \quad (\text{A.11})
\end{aligned}$$

$$\begin{aligned}
&= (\ell+1) \Lambda_0^Q \sum_{\substack{\{0 \leq n'_j \leq N-1\} \\ \sum n'_j = nN - \ell N}} \omega^{-\sum_j j n'_j} K_{\ell N + N + Q}(\{n'_j\}) |\{n'_j\}\rangle \\
&= (\ell+1) \omega^Q \Lambda_0^Q \bar{\mathbf{C}}_0^{(N+\ell N+Q)} \bar{\mathbf{B}}_1^{(nN+Q)} |\Omega\rangle. \quad (\text{A.12})
\end{aligned}$$

In (A.11), (III.19) is again be used to arrive at (A.12). This proves that (86) holds for  $m = \ell + 1$ , and therefore holds for all  $m$ .

## Appendix B. Serre Relation for Special Cases

Let  $\ell = 1$  in (A.8), and multiply it by  $\mathbf{x}_{1,Q}^-$  and then use (A.2), to obtain

$$\begin{aligned}
& (\Lambda_0^Q)^2 \mathbf{x}_{1,Q}^- (\mathbf{x}_{0,Q}^+) (\mathbf{x}_{1,Q}^-)^n |\Omega\rangle = n! \sum_{\substack{\{0 \leq \lambda_j, \mu_j, n'_j \leq N-1\} \\ \sum n'_j = nN, \\ \sum \lambda_j = N+Q, \sum \mu_j = Q}} \omega^{-\sum_j j n'_j} I_{nN-N}(\{n'_j + \mu_j\}; \{\lambda_j\}) \\
&\quad \times \prod_{j=1}^L \begin{bmatrix} n'_j + \mu_j \\ \mu_j \end{bmatrix} \omega^{\mu_j N'_j} |\{n'_j\}\rangle. \quad (\text{B.1})
\end{aligned}$$

Using (III.22) and  $\bar{I}_0(\{\lambda_j\}; \{\mu_j\}) = 1$ , we find

$$I_{nN-N}(\{\mu_j + n'_j\}; \{\lambda_j\}) = 1 + (n-1) \bar{I}_N(\{\lambda_j\}; \{\mu_j + n'_j\}). \quad (\text{B.2})$$

Similar to the derivation of (A.11) from (A.9), we use (A.10), changing variables  $\mu'_j = n_j + \mu_j$  and  $\lambda_j = \lambda'_j + n_j$ , to find

$$\begin{aligned}
& \sum_{\substack{\{0 \leq \mu_j, \lambda_j \leq N-1\} \\ \sum \lambda_j = N+Q, \sum \mu_j = Q}} \bar{I}_N(\{\lambda_j\}; \{n'_j + \mu_j\}) \prod_{j=1}^L \begin{bmatrix} n'_j + \mu_j \\ \mu_j \end{bmatrix} \omega^{\mu_j N'_j} \\
&= \sum_{\substack{\{0 \leq \mu'_j, \lambda'_j \leq N-1\} \\ \sum \lambda'_j = Q, \sum \mu'_j = N+Q}} I_N(\{\mu'_j\}; \{\lambda'_j\}) \prod_{j=1}^L \begin{bmatrix} n'_j + \mu'_j \\ \mu'_j \end{bmatrix} \omega^{\mu'_j N'_j} = \Lambda_0^Q K_{N+Q}(\{n'_j\}), \quad (\text{B.3})
\end{aligned}$$

where (III.19) and (III.4) have been used. Substituting (B.2) into (B.1), and using (B.3) and (III.19), we find

$$\begin{aligned}
\mathbf{x}_{1,Q}^-(\mathbf{x}_{0,Q}^+)(\mathbf{x}_{1,Q}^-)^n|\Omega\rangle &= \frac{n!}{(\Lambda_0^Q)^2} \sum_{\substack{\{0 \leq n'_j \leq N-1\} \\ \sum n'_j = nN}} \omega^{-\sum_j j n'_j} \left[ \Lambda_1^Q K_Q(\{n'_j\}) \right. \\
&\quad \left. + (n-1)\Lambda_0^Q K_{N+Q}(\{n'_j\}) \right] |\{n'_j\}\rangle \\
&= \left[ \frac{\Lambda_1^Q}{\Lambda_0^Q} (\mathbf{x}_{1,Q}^-)^n + \frac{(n-1)}{(n+1)} (\mathbf{x}_{0,Q}^+)(\mathbf{x}_{1,Q}^-)^{n+1} \right] |\Omega\rangle, \tag{B.4}
\end{aligned}$$

where (A.8) is used to get the second equation. Multiplying by  $\mathbf{x}_{1,Q}^-$  on both sides of this equation, we find

$$\begin{aligned}
(\mathbf{x}_{1,Q}^-)^2(\mathbf{x}_{0,Q}^+)(\mathbf{x}_{1,Q}^-)^n|\Omega\rangle &= \left[ \frac{\Lambda_1^Q}{\Lambda_0^Q} (\mathbf{x}_{1,Q}^-)^{n+1} + \frac{(n-1)}{(n+1)} \mathbf{x}_{1,Q}^-(\mathbf{x}_{0,Q}^+)(\mathbf{x}_{1,Q}^-)^{n+1} \right] |\Omega\rangle \\
&= \left[ \frac{2n}{n+1} \frac{\Lambda_1^Q}{\Lambda_0^Q} (\mathbf{x}_{1,Q}^-)^{n+1} + \frac{n(n-1)}{(n+1)(n+2)} (\mathbf{x}_{0,Q}^+)(\mathbf{x}_{1,Q}^-)^{n+2} \right] |\Omega\rangle, \tag{B.5}
\end{aligned}$$

where (B.4) is used again for the second term, and the coefficients are sorted. Similarly, we can show

$$(\mathbf{x}_{1,Q}^-)^3(\mathbf{x}_{0,Q}^+)(\mathbf{x}_{1,Q}^-)^n|\Omega\rangle = \left[ \frac{3n}{n+2} \frac{\Lambda_1^Q}{\Lambda_0^Q} (\mathbf{x}_{1,Q}^-)^{n+2} + \frac{n(n-1)}{(n+2)(n+3)} \mathbf{x}_{0,Q}^+(\mathbf{x}_{1,Q}^-)^{n+3} \right] |\Omega\rangle. \tag{B.6}$$

By substituting (B.4), (B.5) and (B.6) into the second member of the following equation and collecting terms we find

$$\begin{aligned}
[[[\mathbf{x}_{0,Q}^+, \mathbf{x}_{1,Q}^-], \mathbf{x}_{1,Q}^-] \mathbf{x}_{1,Q}^-] (\mathbf{x}_{1,Q}^-)^n |\Omega\rangle &= [\mathbf{x}_{0,Q}^+(\mathbf{x}_{1,Q}^-)^{3+n} - 3\mathbf{x}_{1,Q}^-(\mathbf{x}_{0,Q}^+(\mathbf{x}_{1,Q}^-)^{2+n} \\
&\quad + 3(\mathbf{x}_{1,Q}^-)^2 \mathbf{x}_{0,Q}^+(\mathbf{x}_{1,Q}^-)^{1+n} - (\mathbf{x}_{1,Q}^-)^3 \mathbf{x}_{0,Q}^+(\mathbf{x}_{1,Q}^-)^n] |\Omega\rangle = 0. \tag{B.7}
\end{aligned}$$

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